Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 174 (2007) 79–94

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Topological Perspective on the Hybrid Proof Rules

Balder ten Cate[1]

ISLA – Informatics Institute
Universiteit van Amsterdam
Amsterdam, The Netherlands

Tadeusz Litak[2,3]

School of Information Science, JAIST
Asahidai 1-1, Nomi-shi, Ishikawa
923-1292 Japan

Abstract

We consider the non-orthodox proof rules of hybrid logic from the viewpoint of topological semantics. Topological semantics is more general than Kripke semantics. We show that the hybrid proof rule **BG** is topologically not sound. Indeed, among all topological spaces the **BG** rule characterizes those that can be represented as a Kripke frame (i.e., the Alexandroff spaces). We also demonstrate that, when the **BG** rule is dropped and only the **Name** rule is kept, one can prove a general topological completeness result for hybrid logics axiomatized by pure formulas. Finally, we indicate some limitations of the topological expressive power of pure formulas. All results generalize to neighborhood frames.

Keywords: hybrid logic, proof rules, topological spaces

1 Introduction

In many completeness results for hybrid logics, an important role is played by *non-orthodox rules*, i.e., proof rules involving syntactic side-conditions, also known as *Gabbay-style rules*. For instance, such rules are necessary in order to obtain general Kripke completeness for hybrid logics axiomatized by pure formulas [3]. Various

[1] Email: balder.tencate@uva.nl

[2] Email: litak@jaist.ac.jp

[3] Thanks are due to Eric Pacuit for his comments and to Guram Bezhanishvili, who shared some yet unpublished papers with the second author. Most of all, we would like to express our gratitude to Patrick Blackburn for agreeing to present the paper on our behalf, when it turned out that neither of us could attend the workshop. We are obliged to the organizers for agreeing to this non-standard solution (which failed in the end for reasons beyond control of any logician). Finally, we are obliged to Katsuhiko Sano for careful proof-reading of the final version.

doi:10.1016/j.entcs.2006.11.027

formulations of these non-orthodox proof rules for hybrid logic have been used in the literature under names such as *COV* [9], *Name* and *Paste* [4] or *BG* [3]. Here, we will consider the following formulation of the rules in the language with the global modality:

Name If $\vdash i \to \phi$ then $\vdash \phi$, for i not occurring in ϕ.

BG If $\vdash \mathsf{E}(i \wedge \Diamond j) \to \mathsf{E}(j \wedge \phi)$ then $\vdash \mathsf{E}(i \wedge \Box\phi)$ for $i \neq j$ and j not occurring in ϕ.

The **Name** rule is very natural. It expresses that whenever a formula ϕ is falsifiable, one can name the falsifying world by a fresh nominal. The **BG** rule also seems natural at first. Under the usual Kripke semantics, it expresses that whenever $\Diamond\phi$ holds at a world, one can pick a fresh nominal to name a witnessing ϕ-successor. Under the topological semantics, however, where $\Box\phi$ expresses that ϕ holds throughout some open neighborhood around the current point, it is not clear why the **BG** rule would preserve validity.

This observation forms the starting point of this paper. First, we demonstrate that the **BG** rule fails to preserve validity on many topological spaces, including the real line (Theorem 3.2). In fact, we show (Theorem 3.4) that the **BG** rule characterizes those topological spaces that can be represented as Kripke frames (i.e., the Alexandroff spaces).

Next, we prove that the **BG** rule can be eliminated from the axiomatization without sacrificing completeness: every hybrid logic extending **S4** by means of pure formulas, *without the* **BG** *rule*, is complete with respect to some class of topological spaces (Theorem 4.1). These results generalize to neighborhood frames (Section 6).

Of course, pure formulas have rather limited expressive power: while the separation axioms T_0 and T_1 can be defined by means of pure formulas, many other properties of topological spaces cannot be defined in this way. In fact, in the absence of **BG** even the hybrid variant of **S4** cannot be axiomatized with pure formulas (Corollary 6.4). Nevertheless, the construction used in our completeness proof seems to work for an interesting class of axioms, not all of which are topologically equivalent to pure formulas. It is left as an open question how large exactly this class is.

Concerning the main question of the workshop — *what is the proper way to hybridize a logic?* — we can conclude that a crucial decision lies in the choice of non-orthodox rules. This decision in turn should depend on whether the intended semantics is topological or graph-like in nature. Whichever choice you make, don't worry: hybrid proof mechanisms are going to cater for your needs. There are many paths to hybrid paradise.

2 Background

2.1 Hybrid logic and hybrid proof rules

The hybrid language we will consider in this paper $\mathcal{H}(\mathsf{E})$ is obtained by enriching ordinary modal logic with *nominals*, a second sort of atomic formula, typically written i, j, k,..., and with the *global modality* E. More precisely, given a countable set of ordinary proposition letters PROP and a countable set of nominals NOM, we define the formulas of $\mathcal{H}(\mathsf{E})$ to be

$$\phi \ ::= \ p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid \mathsf{E}\phi$$

where $p \in$ PROP and $i \in$ NOM. Other connectives are introduced as abbreviations. In particular, $\Box$ is understood thorough the paper as $\neg\Diamond\neg$. Nevertheless, later on it will be sometimes more convenient to introduce some notions treating $\Box$ as the basic connective. $Sub(\phi)$, *the subformula closure* of ϕ is the set of all subformulas of ϕ. *A substitution instance of* ϕ is defined as usual in propositional logic with the additional requirement that nominals can be replaced only by nominals. Nominals act as proposition letters, except that their valuation is required to be a singleton set. Thus, for example, the hybrid formula $\Diamond(i \wedge p) \wedge \Diamond(i \wedge \neg p)$ is unsatisfiable due to the fact that i is a nominal. The global modality allows us to express that a formula holds *somewhere* in the model: $\mathsf{E}\phi$ is true at a point w iff there is a point v (not necessarily related to w) satisfying ϕ.

The nominals and the global modality make it possible to define properties that are not definable in the basic modal language. For instance, no formula of the basic modal language can define the class of irreflexive Kripke frames, but it is easy to see that $i \to \neg\Diamond i$ is valid on a Kripke frame iff the frame is irreflexive. Note that *valid* here means true at all worlds in the frame under all valuations that make nominals true at a unique world. Observe that this formula contains no proposition letters, only nominals. We call such formulas *pure*.

One of the great merits of hybrid logic is that there is a general completeness result for logics axiomatized by pure formulas. This result relies on the use of non-orthodox proof rules: rules that involve syntactic side-conditions. The full axiomatization of minimal logic in $\mathcal{H}(\mathsf{E})$ is given in Table 1. For any set of formulas Γ, define $\mathbf{K}^{\mathbf{Name},\mathbf{BG}}_{\mathcal{H}(\mathsf{E})}\Gamma$, to be the smallest set of formulas containing Γ and closed under all the axioms and rules from Table 1.

Theorem 2.1 *Let Γ be any set of pure formulas. Then $\mathbf{K}^{\mathbf{Name},\mathbf{BG}}_{\mathcal{H}(\mathsf{E})}\Gamma$ is strongly complete for the class of Kripke frames defined by Γ.*

This result crucially depends on the use of non-orthodox proof rules: every axiomatization of which each pure extension is complete must contain either infinitely many proof rules or non-orthodox ones. [3]

Table 1
Axioms and rules for $\mathcal{H}(\mathsf{E})$

CT	$\vdash \phi$, for all classical tautologies ϕ
K	$\vdash \Box(p \to q) \to \Box p \to \Box q$
$\mathbf{K}_{\mathsf{A}}$	$\vdash \mathsf{A}(p \to q) \to \mathsf{A}p \to \mathsf{A}q$
$\mathbf{Ref}_{\mathsf{E}}$	$\vdash p \to \mathsf{E}p$
$\mathbf{Trans}_{\mathsf{E}}$	$\vdash \mathsf{EE}p \to \mathsf{E}p$
$\mathbf{Sym}_{\mathsf{E}}$	$\vdash p \to \mathsf{AE}p$
$\mathbf{Incl}_{\Diamond}$	$\vdash \Diamond p \to \mathsf{E}p$
$\mathbf{Incl}_{i}$	$\vdash \mathsf{E}i$
Nom	$\vdash \mathsf{E}(i \wedge p) \to \mathsf{A}(i \to p)$
MP	If $\vdash \phi \to \psi$ and $\vdash \phi$ then $\vdash \psi$
Nec	If $\vdash \phi$ then $\vdash \Box\phi$
$\mathbf{Nec}_{\mathsf{A}}$	If $\vdash \phi$ then $\vdash \mathsf{A}\phi$
Subst	If $\vdash \phi$ then $\vdash \phi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals.
Name	If $\vdash i \to \phi$ then $\vdash \phi$, for i not occurring in ϕ.
BG	If $\vdash \mathsf{E}(i \wedge \Diamond j) \to \mathsf{E}(j \wedge \phi)$ then $\vdash \mathsf{E}(i \wedge \Box\phi)$, for $i \neq j$ and j not occurring in ϕ.

2.2 Topological Semantics

Topological spaces are usually defined as pairs $\langle T, O\rangle$, consisting of a non-empty set T ("the points"), and a set O of subsets of T ("the open sets"), such that $\emptyset, T \in O$ and such that O is closed under arbitrary unions and finite intersections. For present purposes, it is convenient to use a slightly different presentation.

Definition 2.2 A topological space is a pair $\langle T, \tau\rangle$, consisting of a non-empty set T, and a function τ that assigns to each $x \in T$ a set of subsets of T ("the basic open neighborhoods around x") satisfying the following properties:

non-emptiness there is some $X \in \tau(x)$,

filter $X, Y \in \tau(x)$ iff $X \cap Y \in \tau(x)$,

T for all $X \in \tau(x), x \in X$,

4 for all $X \in \tau(x), \{y \in T \mid X \in \tau(y)\} \in \tau(x)$.

In presence of the **filter** condition, **non-emptiness** is equivalent to the requirement that $T \in \tau(x)$. For every $x \in T$, $\tau(x)$ is called *the neighborhood base of* x. For any $X \subseteq T$, we define the *interior* of X, $\Box_\tau X$, to be the set $\{x \in T \mid X \in \tau(x)\}$. Thus, **4** says that for all $X \in \tau(x), \Box_\tau X \in \tau(x)$.

It will become clear in Section 6 why we use this definition of topological spaces. The two versions are equivalent, though (cf. any textbook on set-theoretic topology):

Fact 2.3 *Given any topological space* $\mathfrak{T} := \langle T, \tau\rangle$ *in the sense of Definition 2.2, the family of open sets* $O_\tau := \{\Box_\tau X \mid X \subseteq T\}$ *contains* $\emptyset$ *and* T *and is closed under arbitrary unions and finite intersections. Conversely, if* $O \subseteq \wp(T)$ *contains* $\emptyset$ *and* T *and is closed under arbitrary union and finite intersections, then*

$$\langle T, \tau_O : T \ni x \mapsto \{X \subseteq T \mid \exists Y \in O \text{ such that } x \in Y \text{ and } Y \subseteq X\}) \subseteq \wp(T)\rangle$$

is a topological space in the sense of Definition 2.2. Moreover, $\langle T, \tau_{O_\tau} \rangle$ is the same topological space as $\langle T, \tau \rangle$.

A *topological model* $\langle \mathfrak{T}, V \rangle$ consists of a topological space $\mathfrak{T}$ and a valuation function V assigning subsets of T to propositional variables and elements of T to nominals. Satisfaction of a formula ϕ at a point x in a topological model $\langle \mathfrak{T}, V \rangle$ (notation: $\langle \mathfrak{T}, V \rangle, x \models \phi$) is defined as follows:

$$\begin{array}{ll}
\mathfrak{T}, V, x \models p & \text{iff } x \in V(p) \\
\mathfrak{T}, V, x \models i & \text{iff } x = V(i) \\
\mathfrak{T}, V, x \models \neg\phi & \text{iff } \mathfrak{T}, V, x \not\models \phi \\
\mathfrak{T}, V, x \models \phi \wedge \psi & \text{iff } \mathfrak{T}, V, x \models \phi \text{ and } \mathfrak{T}, V, x \models \psi \\
\mathfrak{T}, V, x \models \Box\phi & \text{iff } x \in \Box_\tau \{y \in T \mid \mathfrak{T}, V, y \models \phi\} \\
\mathfrak{T}, V, x \models \mathsf{A}\phi & \text{iff for every } y \in T,\ \mathfrak{T}, V, y \models \phi
\end{array}$$

$V(\phi)$, as usual, stands for denotation of ϕ under V. This sometimes creates slight ambiguity: $V(i)$ can stand either for the element assigned to i by V or for the corresponding singleton. We will be rather careless here, as it is not likely to create any confusion.

It is straightforward to see that the **S4** axioms $\Box p \to p$ and $\Box p \to \Box\Box p$ hold regardless of the valuation (i.e., are valid). Since the work of McKinsey and Tarski [14], it is known that in the basic modal language (i.e., without universal modality and nominals) **S4** is not only *sound* but also *complete* with respect to topological spaces. Before we prove a more general version of this theorem for hybrid logic, let us recall why adding nominals is an attractive step from topological perspective.

Example 2.4 The following properties are definable in $\mathcal{H}(\mathsf{E})$:

- T_0, i.e., for all $x \neq y$, either there is an $X \in \tau(x)$ such that $y \notin X$, or there is an $X \in \tau(y)$ such that $x \notin X$

  is defined by $\mathsf{E}(i \wedge \neg j) \to \mathsf{E}(i \wedge \Box\neg j) \vee \mathsf{E}(j \wedge \Box\neg i)$

- T_1, i.e., for all $x \neq y$, there is an $X \in \tau(x)$ such that $y \notin X$

  is defined by $\Diamond i \to i$

- density-in-itself, i.e., for every x, $\{x\} \notin \tau(x)$

  is defined by $\Diamond\neg i$

These properties are not definable in the basic modal language [14,5].

We mentioned earlier that topological spaces generalize Kripke frames. Every reflexive and transitive (that is, *quasi-ordered*) Kripke frame $\mathfrak{F} = \langle W, R \rangle$ gives rise to a topological space: simply take $\tau_R(x) = \{X \subseteq W \mid \forall y(xRy \to y \in X)\}$. Not every topological space can be represented by a Kripke frame in this way. In fact, a space is representable by means of a Kripke frame iff every $\tau(x)$ contains a smallest element. Such spaces are called *Alexandroff spaces*.

Fact 2.5 *A space is Alexandroff iff for arbitrary family* $\{X_i\}_{i\in I}$ *of subsets of* I, $\Box_\tau \bigcap_{i\in I} X_i = \bigcap_{i\in I} \Box_\tau X_i$ *or, dually,* $\Diamond_\tau \bigcup_{i\in I} X_i = \bigcup_{i\in I} \Diamond_\tau X_i$.

There is in fact a category-theoretical equivalence between quasi-orders and Alexandroff spaces (cf. [8,16]), but we will not go into the details here.

3 Hybrid proof rules from topological perspective

Topological semantics for hybrid logic has received some attention in recent literature, where it is noticed that several modally undefinable properties of topological spaces are definable using nominals. However, not much is known about axiomatics for hybrid logic under the topological semantics. It is not hard to see that all axioms of hybrid logic are also sound under the topological interpretation. The **Name** rule is also topologically sound:

Theorem 3.1 *The* **Name** *rule preserves validity on every topological space.*

Proof. If ϕ is falsified on a topological space $\mathfrak{T}$ at a point x, then $i \to \phi$ is falsified under the same valuation extended by sending i to x. □

The **BG** rule, on the other hand, is topologically dangerous even on the most well known topology, the real line, as the following shows:

Theorem 3.2 *The* **BG** *rule does not preserve validity on the real line, and, indeed, on any non-discrete* T_1 *space.*

Proof. By Example 2.4, $\Diamond i \to i$ defines the T_1 separation property: for all $x \neq y$, there is an open around x to which y does not belong. Clearly, this formula is valid on the real line. Using the axioms of hybrid logic, and classical modal inference rules — which are all topologically sound — we can derive from this

$$\mathsf{E}(j \wedge \Diamond i) \to \mathsf{E}(i \wedge (\mathsf{E}(j \wedge p) \to p))$$

This indeed defines the same class as $\Diamond i \to i$. A single application of the **BG** rule now yields

$$\mathsf{E}(j \wedge \Box(\mathsf{E}(j \wedge p) \to p))$$

This formula defines the same class of topological spaces as $p \to \Box p$, namely the class of all spaces with *discrete topology*, i.e., topology in which the set of opens is the full powerset. To see it, assume that $\mathfrak{T}, V, x \not\models p \to \Box p$. It is enough to set $V(j) := x$ to refute the above formula. Conversely, assume $\mathsf{E}(j \wedge \Box(\mathsf{E}(j \wedge p) \to p))$ is refuted by V. It can happen only if $V(j) \in V(p)$ — otherwise $\mathsf{E}(j \wedge p) \to p$ has a false antecedent and hence is true at every point in the space — and yet $V(j) \notin \Box_\tau V(p)$. But it means $p \to \Box p$ is refuted. Alternatively, one can prove syntactically that the two formulas are equivalent using the **Name** rule, which is topologically sound by Theorem 3.1.

$\mathbb{R} \nvDash p \to \Box p$ and the conclusion follows. □

Incidentally, by the same argument, almost all interesting topological hybrid logics are Kripke incomplete:

Corollary 3.3 *The hybrid logic of the real line, and in fact the hybrid logic of any class of T_1 spaces containing a non-discrete space, is Kripke incomplete.*

Proof. These logics are not closed under the **BG** rule, whereas every Kripke complete logic is. □

Not surprisingly, the **BG** rule *is* sound for Alexandroff topologies, which can be represented as Kripke frames. In fact, it turns out that the **BG** rule characterizes exactly the Alexandroff topologies. Let us say that a space *admits* **BG** if every valuation falsifying the consequent of the **BG** rule can be extended to a valuation falsifying the antecedent.[4]

Theorem 3.4 *A space is Alexandroff iff it admits* **BG**.

Proof. Assume the space is Alexandroff and let $\mathfrak{T}, V \nvDash \mathsf{E}(i \wedge \Box\phi)$. Then there is some y which belongs to the smallest element of $\tau(V(i))$ and does not belong to $V(\phi)$. Define $V'(j) := y$, valuations of all other variables unchanged. It follows that $V'(i) \in V'(\Diamond j)$ and not $j \in V'(\phi)$. Thus, the antecedent of the rule is falsified.

On the other hand, suppose the space is non-Alexandroff. Then there is a point x s.t. $\tau(x)$ contains no smallest element. Define $f : \tau(x) \ni X \mapsto \{y \in X \mid \exists X' \in \tau(x). y \notin X'\} \in \wp(T)$. This is a sequence of sets indexed by elements of $\tau(x)$ whose all elements are non-empty and hence — by the Axiom of Choice — it has a choice function $\tau(x) \ni X \mapsto g(X) \in T$. We may think of $\{g(X)\}_{X \in \tau(x)}$ as of a sequence of elements approximating x s.t. (1) for every $X \in \tau(x)$, $x \notin \Diamond_\tau\{g(X)\}$ and yet (2) $x \in \Diamond_\tau g[\tau(x)]$. Define $V(i) := x$, $V(p) := T - g[\tau(x)]$ (complement of the range of g). Then by (2) $\mathfrak{T}, V, x \nvDash \mathsf{E}(i \wedge \Box p)$ but by (1) for no valuation V' agreeing with V on i and p it is the case that $\mathfrak{T}, V', x \nvDash \mathsf{E}(i \wedge \Diamond j) \to \mathsf{E}(j \wedge p)$. □

Alexandroff spaces do not form a particularly interesting class of spaces, and therefore, from a topological perspective, the **BG** rule is rather *ad hoc*. Inspired by Theorem 3.4, one could consider variations of the rule. For instance, it can be naturally weakened as follows:

$$\mathbf{BG}' \text{ From } \mathsf{E}(i \wedge \Diamond j) \to \mathsf{E}(j \wedge \Diamond\phi) \text{ for } j \notin \mathit{Sub}(\phi), \text{ infer } \mathsf{E}(i \wedge \Box\Diamond\phi)$$

Indeed, this new rule turns out to define a strictly weaker property than that of Alexandroffness.

Theorem 3.5 *All Alexandroff spaces admit the* **BG**$'$ *rule but not vice versa. Not every space admits* **BG**$'$.

[4] This definition is inspired somewhat by similar notions used in [3,17]. An alternative would be to require only that the logic of the class in question is closed under the **BG** rule. However, under this weaker notion of admittance, rules such as the **BG** rule are not likely to characterize any interesting semantic property.

Proof. One direction of the first claim is trivial. The other direction: consider the real numbers and as non-trivial open sets take all intervals (r, r') with $r < 0 < r'$. This space is easily seen to admit $\mathbf{BG}'$. It fails to be Alexandroff though, as the element 0 has no smallest open neighborhood.

To see that not every space admits $\mathbf{BG}'$, consider the real line with standard topology. The element 0 is in the closure of the open interval $(0, 1)$ but it is not in the closure of any singleton from this interval. $\square$

Still, topologically the most natural move is to drop the **BG** rule completely. As the next section shows, this is indeed a feasible option. Every hybrid logic axiomatized by pure formulas is topologically complete *without the* **BG** *rule*.

Remark 3.6 Using the techniques of [6], it can be shown that hybrid logics axiomatized with modal Sahlqvist formulas are all Kripke complete. In case of extensions of $\mathbf{S4}_{\mathcal{H}(\mathsf{E})}$, this entails completeness with respect to a class of Alexandroff topologies, *even without the* **Name** *and* **BG** *rules*. It follows by Theorem 3.1 that the **Name** rule is admissible, and by Theorem 3.4 that the **BG** rule is admissible in these logics.

4 Completeness of pure extensions

In this section, we show that all hybrid logics axiomatized by pure formulas are topologically complete. This generalizes known results for Kripke semantics. The most important difference with these known results is that, in the topological semantics, we cannot make use of the **BG** rule, as it is topologically unsound. Instead, we will only use the **Name** rule. Recall that $\mathbf{K}^{\mathbf{Name},\mathbf{BG}}_{\mathcal{H}(\mathsf{E})}\Gamma$, was defined to be the smallest set of formulas containing Γ and closed under all the axioms and rules from Table 1. $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$ is defined similarly, but we do not require closure under **BG**. We will use $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$ as a shorthand for $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}(\Gamma \cup \{\Box p \to p, \Box p \to \Box\Box p\})$.

We say that a hybrid logic $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$ is *strongly complete* with respect to a class K of topological spaces if every $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$-consistent set of formulas can be jointly satisfied (at some point, under some valuation) on a space in K. Then, the main result of this section is the following:

Theorem 4.1 *For every set Γ of pure $\mathcal{H}(\mathsf{E})$ formulas, $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$ is strongly complete with respect to the class of topological spaces defined by Γ.*

We will make use of the following variant of The Lindenbaum Lemma [12].

Lemma 4.2 *Let Γ be any set of hybrid formulas. Every $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$-consistent set of formulas Δ can be extended to a $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$-consistent set of formulas Δ^+ satisfying the following conditions:*

- *Δ^+ contains all formulas from $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$ and a fresh nominal j,*
- *for every ϕ, exactly one of $\{\phi, \neg\phi\}$ belongs to Δ^+,*

- *for every* $\mathsf{E}\psi \in \Delta^+$*, there is a nominal* i *which does not appear in* ψ *such that* $\mathsf{E}(i \wedge \psi) \in \Delta^+$*.*

The last condition could be baptized *weak namedness.* We may also say that Δ^+ is *pasted for* E*-modality.*

The proof of Theorem 4.1 we will give resembles the well-known Henkin construction in first-order completeness proofs.

Proof of Theorem 4.1. Let Γ be any set of pure formulas, and Δ a $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$-consistent set of formulas. Let Δ^+ be obtained from Δ using Lemma 4.2. We will construct a topological model $\mathcal{T} := (T, \tau, V)$ out of equivalence classes of nominals in Δ^+.

As the universe T , we pick the set of all $\equiv_{\Delta^+}$-equivalence classes of nominals, where $i \equiv_{\Delta^+} j$ iff $\mathsf{E}(i \wedge j) \in \Delta^+$. The reader may check that this is indeed an equivalence relation.

For every $[i] \in T$, we define $\tau([i])$ as the set of all $X \subseteq T$ for which there exists a formula ϕ such that $\mathsf{E}(i \wedge \Box\phi) \in \Delta^+$ and $\{[j] \in T \mid \mathsf{E}(j \wedge \phi) \in \Delta^+\} \subseteq X$. In other words, X is open if every point of X has an open neighborhood contained in X: namely, a suitable neighborhood of the equivalence class of i is defined by some $\Box\phi$ s.t. according to Δ^+ (a) $\Box\phi$ holds at i and (b) every equivalence class where ϕ holds belongs to X.

The valuation V is defined by letting $V(p) := \{[i] \in T \mid \mathsf{E}(i \wedge p) \in \Delta^+\}$ and $V(i) := [i]$.

Claim 4.3 τ *satisfies the* ***non-emptiness*** *and* ***filter*** *conditions.*

Proof. It follows from the definition of τ that, for every $[i]$, $T \in \tau([i])$. Next, assume that $X, Y \in \tau([i])$. It means there are appropriate formulas ϕ_X and ϕ_Y witnessing the membership. But then $\phi_X \wedge \phi_Y$ shows that $X \cap Y \in \tau([i])$. $\Box$

Claim 4.4 τ *satisfies conditions* **T** *and* **4***.*

Proof. Assume $[i] \in \Box_\tau X$. This means that there is a suitable ϕ_X s.t. $\mathsf{E}(i \wedge \Box\phi_X) \in \Delta^+$. Now:

(i) Because of the **T** axiom and the assumption on ϕ_X, it means $[i] \in X$, thus $\Box_\tau X \subseteq X$.

(ii) To show that $\Box_\tau X \subseteq \Box_\tau\Box_\tau X$, it is enough to set $\phi_{\Box X} := \Box\phi_X$ and use the axiom **4**.

$\Box$

Claim 4.5 (Truth) *For every* ϕ *and* i*,* $\mathfrak{T}, [i] \models \phi$ *iff* $\mathsf{E}(i \wedge \phi) \in \Delta^+$*.*

Proof. By induction. The interesting clause is the one for $\Box$. It follows from the definition of τ that $\mathfrak{T}, [i] \models \Box\phi$ holds iff

> there is ψ such that $\mathsf{E}(i \wedge \Box\psi) \in \Delta^+$ and for every j, $\mathsf{E}(j \wedge \psi) \in \Delta^+$ implies $\mathfrak{T}, [j] \models \phi$

Using the induction hypothesis and the weak namedness of Δ^+ (cf. Lemma 4.2) one can show that this holds iff there is ψ such that $\mathsf{E}(i \wedge \Box\psi) \in \Delta^+$ and $\mathsf{A}(\psi \to \phi) \in \Delta^+$. Clearly, the latter holds iff $\mathsf{E}(i \wedge \Box\phi) \in \Delta^+$ (in one direction, pick $\psi = \phi$, and in the other direction, use the $\mathbf{K}_{\mathsf{A}}$ and $\mathbf{Incl}_{\Diamond}$ axioms). The clause for A also requires weak namedness; we leave the details to the reader. □

In particular, it follows that the point $[j]$ satisfies all formulas in Δ.

All that remains is to check that all substitution instances of formulas in Γ hold at every point in $\mathfrak{T}$. But each such substitution instance ϕ belongs to the logic, and hence, by the $\mathbf{Nec}_{\mathsf{A}}$ rule and the fact that Δ^+ is an MCS, $\mathsf{A}\phi \in \Delta^+$, and hence, by Claim 4.5, $\mathfrak{T} \models \phi$. □

In Section 7, we briefly discuss how this result can be extended to other hybrid languages.

This proof is a hybrid of the Henkin-style technique of *named models* and the original McKinsey-Tarski topological completeness result for **S4** in the basic modal language [14]. Aiello et al. [1] give a proof analogous to the one of McKinsey and Tarski, but formulated directly in terms of MCS's rather than in the language of algebra. The topological space $\mathfrak{T}_L := \langle T_L, \tau_L \rangle$ is constructed out of MCS's in the basic modal language without any non-standard rules. For every such MCS Γ, $\tau_L(\Gamma) := \{X \subseteq T_L \mid \{\Delta \mid \Box\phi \in \Delta\} \subseteq X$ for some $\Box\phi \in \Gamma\}$. Both our proof and the one for the basic modal language take 'to be a base open set' to mean 'to be an extension of some $\Box\phi$', but here instead of dealing with *all MCS's*, we work only with equivalence classes of *nominals* — and, in addition, this equivalence relation is defined with respect to a *single* MCS. The relationship of our named topological models with those canonical topological ones can be compared then to the one between named Kripke models (in the language with the **BG** rule) and the standard canonical model construction.

Remark 4.6 $\mathfrak{T}_L$ is a subtopology of the Stone topology on all MCS's and hence must be, for example, dense-in-itself (see, e.g., [1]). However, as follows from Example 2.4, density-in-itself is definable in $\mathcal{H}(\mathsf{E})$. Hence, the hybrid logic of spaces which are dense-in-itself is a proper extension of $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}$. This alone shows that topology defined in the proof of Theorem 4.1 has different properties than $\mathfrak{T}_L$ and is not always homeomorphic to it.

Remark 4.7 Theorem 5.2 in Bezhanishvili et al. [2] suggests that the McKinsey-Tarski construction in the basic modal language does not work for proper extensions of **S4**. The authors introduce *topo-canonical extensions* of arbitrary closure algebras (similar to *canonical extensions*, which generalize the canonical model construction to arbitrary, not necessarily free, modal algebras), and show that these topo-canonical extensions do not preserve any equation corresponding to an axiom properly extending **S4**.

This does not yet imply that the McKinsey-Tarski construction must fail for all extensions of **S4** in the basic modal language: this would require showing that the topological spaces obtained this way from *free (Lindenbaum-Tarski) closure algebras* do not verify any formulas besides those in **S4**. Still, it seems that the McKinsey-Tarski construction does need the presence of individual names in the syntax to reveal its true potential.

5 Euclidean spaces

Probably the most well known examples of topological spaces are the real line $\mathbb{R}$ and its finite powers $\mathbb{R}^n$ equipped with Euclidean topology. McKinsey and Tarski [14] proved that the modal logic of each of these spaces is **S4**, which is also the modal logic of the class of all topological spaces. In particular, none of the interesting topological properties of the real line can be detected in the basic modal language. In hybrid logic, the situation is quite different: the lower separation properties T_0 and T_1 are definable using nominals and the global modality, and Corollary 3.3 shows that logics of Euclidean spaces are not even Kripke complete.

At present, we do not have a complete axiomatization of the hybrid logic of the real line. However, for $n \geq 2$ Kudinov [10] presents an axiomatization of the logic of $\mathbb{R}^n$ in the language equipped with the *difference modality*. These two languages (viz. the modal language with the difference modality and the hybrid language with the global modality) are closely related. In particular, they have the same expressive power on topological spaces [12]. The axioms proposed by Kudinov can be translated into the hybrid language as follows:

T_1-separation $\Diamond i \to i$,

density-in-itself $\Diamond \neg i$,

connectedness $\mathsf{A}(\Box p \vee \Box \neg p) \to \mathsf{A}p \vee \mathsf{A}\neg p$,

l1-connectedness $i \wedge \Box(\neg i \to \Box q \vee \Box \neg q) \to \Box(\neg i \to q) \vee \Box(\neg i \to \neg q)$

The proof of Kudinov's completeness result has not yet been published. But even without access to the details of this proof, we can apply the correspondence results of [12] in order to derive that the hybrid logic axiomatized by the above four axioms (and with the **Name** rule) must also be complete with respect to the same topological spaces as the logic of Kudinov — assuming, of course, that this logic is topo-complete as intended.

Remark 5.1 In an already published paper, Kudinov [11] proves an intermediate result: that the difference logics of dense-in-itself spaces and dense-in-itself T_1 spaces are axiomatizable by the difference analogues of $\mathcal{H}(\mathsf{E})$-formulas above. As the suitable hybrid formulas are pure, this can be obtained as a corollary of Theorem 4.1, Example 2.4 and translation results of Litak [12]. However, Kudinov's axiomatization does not use any non-standard rules: not even the difference version of **Name**. It is also worth mentioning that for all systems under consideration [11] proves also the finite model property with respect to a certain non-standard semantics.

Note that the two last formulas in the above list are not pure. Indeed, the hybrid logic of $\mathbb{R}^n$, for any $n \in \omega$, cannot be axiomatized using only pure formulas:

Theorem 5.2 *No two dense-in-itself T_1-spaces are distinguishable by means of pure formulas. In particular, connectedness is not definable by means of pure formulas.*

Proof. This follows from a more general result: every pure $\mathcal{H}(\mathsf{E})$ axiom has a correspondent in L_I, the topological language that extends first-order logic with formulas of the form $[I_x\phi](y)$ ("*y belongs to the interior of the set defined by* $\phi(x)$"). Makowsky and Ziegler [13] proved that L_I formulas cannot distinguish dense-in-itself T_1-spaces. See [5] for more information on the relation between hybrid logics and L_I.

We also give a direct proof of Theorem 5.2 here as it is going to be generalized in Section 6.

Let $\mathfrak{T}$ and $\mathfrak{T}'$ be dense-in-itself T_1-spaces, and suppose for the sake of contradiction that $\mathfrak{T} \models \phi$ and $\mathfrak{T}' \not\models \phi$ for some pure formula ϕ. Then there are V' and w' such that $\mathfrak{T}', V', w' \not\models \phi$. Let V be any valuation for $\mathfrak{T}$ that sends any two nominals to the same point iff V' does, and let w be any element of $\mathfrak{T}$ such that w and w' agree on the nominals they satisfy. An induction argument shows that the following facts hold for all pure formulas ψ:

(i) Both in $\langle \mathfrak{T}, V\rangle$ and $\langle \mathfrak{T}', V'\rangle$, ψ holds either at finitely or at cofinitely many points
(ii) $\mathfrak{T}, V \vDash \psi$ iff $\mathfrak{T}', V' \vDash \psi$
(iii) ψ holds at cofinitely many points at $\langle \mathfrak{T}, V\rangle$ iff ψ holds at cofinitely many points at $\langle \mathfrak{T}', V'\rangle$
(iv) ψ holds at finitely many points at $\langle \mathfrak{T}, V\rangle$ iff ψ holds at finitely many points at $\langle \mathfrak{T}', V'\rangle$
(v) $\mathfrak{T}, V \vDash \neg\psi$ iff $\mathfrak{T}', V' \vDash \neg\psi$

For atomic formulas this follows by construction of the valuation. The inductive steps for the Boolean connectives and E are straightforward. For $\Diamond$, we use the fact that in every T_1 dense-in-itself space, $\Diamond_\tau X = \top$ for every cofinite set and $\Diamond_\tau X = X$ for every finite set.

It follows that $\mathfrak{T}, V, w \not\models \phi$, and hence $\mathfrak{T} \not\models \phi$. A contradiction.

For the second part of the result, note that some T_1 dense-in-itself spaces are connected (like $\mathbb{R}$), others are not. Take for example the set $[0,1) \cup [2,3)$ with the topology induced from $\mathbb{R}$. □

Corollary 5.3 *The logic of $\mathbb{R}^n$ for any $n \in \omega$ is not axiomatizable by pure formulas.*

6 Neighborhood semantics

While the topological semantics clearly generalizes Kripke semantics, it does so only for logics above **S4**. In this section, we consider another semantics for modal and hybrid logics, that coincides with the topological semantics for logics above **S4**, but that also applies to logics below **S4**. This semantics is known as *normal neighborhood semantics* or *Scott-Montague semantics*. In this section, we will show that our results can be generalized to this setting.

A *normal neighborhood frame* $\langle T, \tau \rangle$ consists of a non-empty set T and a function $\tau : T \mapsto \wp(\wp(T))$ s.t. for every $x \in T$, $\tau(x) \neq \emptyset$ and $X, Y \in \tau(x)$ iff $X \cap Y \in \tau(x)$. In other words, we require that τ assigns to every element *a filter* over W. This makes the frames under consideration suitable for *normal* modal logics, hence the name we use here. The definitions of *valuations*, *normal neighborhood models*, *satisfaction*, etc., are analogous to those in the topological case.

Neighborhood frames have been particularly useful in a *non-normal* setting, i.e., for logics which do not contain some instance of **K** or are not closed under **Nec**. Remark 6.5 below discusses the possibility of weakening $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}$ along these lines. In the rest of this section however, unless otherwise specified, we will only consider *normal* neighborhood frames, and omit the qualification 'normal'.

A careful inspection of the proof of Theorem 4.1 reveals that there was only one place where we used the **S4** axioms: in the proof of Claim 4.4. It follows that Theorem 4.1 can be generalized to neighborhood semantics, obtaining neighborhood completeness of all hybrid logics axiomatized by pure formulas, *not necessarily extending* **S4**. More precisely,

Theorem 6.1 *For every set of pure $\mathcal{H}(\mathsf{E})$ formulas Γ, $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}\Gamma$ is sound and complete with respect to the class of neighborhood frames defined by Γ.*

Proof. It is just the proof of Theorem 4.1 with Claim 4.4 removed. Details are left to the reader. $\square$

One might wonder if Theorem 4.1 could have been derived as a corollary of Theorem 6.1. Recall that $\mathbf{K}^{\mathbf{Name},\mathbf{BG}}_{\mathcal{H}(\mathsf{E})}\{\Box p \to p, \Box p \to \Box\Box p\} = \mathbf{K}^{\mathbf{Name},\mathbf{BG}}_{\mathcal{H}(\mathsf{E})}\{i \to \Diamond i, \Diamond\Diamond i \to \Diamond i\}$. That is, the class of quasi-orders is definable in the class of all Kripke frames by means of pure formulas. The surprising thing is that this statement is no longer true when *Kripke frames* are replaced by *neighborhood frames* and *quasi-orders* by *topological spaces*. The axiom **T** is unproblematic:

Fact 6.2 *The following are equivalent for every neighborhood frame $\mathfrak{T} := \langle T, \tau \rangle$:*

(i) $\mathfrak{T} \models \Box p \to p$,

(ii) $\mathfrak{T} \models i \to \Diamond i$,

(iii) *For every $x \in T$ and every $X \in \tau(x)$, $x \in X$.*

The situation with the axiom **4** is different, though.

Theorem 6.3 *No two neighborhood frames where $\Diamond i \leftrightarrow i$ and $\Diamond\neg i$ hold are distinguishable by pure formulas.*

Proof. Exactly the same as Theorem 5.2. $\square$

Corollary 6.4 *Neither the condition **4** nor the property of being a topological space are definable by means of pure formulas.*

Proof. Fix an infinite sequence of disjoint infinite subsets of natural numbers $\{A_n\}_{n \in \mathbb{N}}$. For every A_n, there exists a non-principal ultrafilter F_n containing A_n. For every n, let $\tau(n) := \{\{n\} \cup X \mid X \in F_n\}$. For every n, $\Box_\tau \neg\{n\} = \mathbb{N} - \{n\}$

and $\Box_\tau\{n\} = \emptyset$, and thus $\langle \mathbb{N}, \tau\rangle$ validates the same pure formulas as every T_1 dense-in-itself topological space. In particular, it validates $\Diamond i \leftrightarrow i$ and $\Diamond\neg i$. But $\Box_\tau(\{n\} \cup A_n) = \{n\}$ and hence $\Box_\tau\Box_\tau(\{n\} \cup A_n) = \emptyset$. Thus, $\langle \mathbb{N}, \tau\rangle \not\models \Box p \to \Box\Box p$ and $\langle \mathbb{N}, \tau\rangle$ is not a topological space. □

Hence, the construction used in the proof of Theorems 4.1 and 6.1 applies also to important logics which are not axiomatizable by means of pure formulas — like $\mathbf{K4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}$ or $\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}$. As was mentioned in the introduction, it would be interesting to investigate how large exactly this class is.

Remark 6.5 It is possible to generalize this completeness result to a reasonably well-behaved class of hybrid non-normal logics with the universal modality. Dropping or weakening **K** and/or **Nec** on the syntactic side would simply correspond to dropping or weakening **non-emptiness** and **filter** conditions on the semantic side. In the language with E, if we do not want non-normal frames to satisfy the **non-emptiness** condition, we also have to drop $\mathbf{Incl}_\Diamond$. Other axioms and rules can remain unchanged, as they do not involve $\Diamond$. Now, careful analysis of the proof of Theorem 4.1 reveals that Claim 4.5 uses the following theorem of $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}$:

$$\mathsf{E}(p \wedge \Box q) \wedge \mathsf{A}(q \to r) \to \mathsf{E}(p \wedge \Box r),$$

which on the basis of axioms for E is equivalent to

$$\mathsf{A}(q \to r) \to \mathsf{A}(\Box q \to \Box r)$$

(substitute $\neg\Box r$ for p, the converse direction is left as an exercise) and this in turn is equivalent to

$$\mathbf{M}_{\mathsf{A}} \quad \mathsf{A}(q \to r) \to (\Box q \to \Box r).$$

This axiom forces closure under the rule

$$\mathbf{Mon} \quad \text{If} \vdash \psi \to \chi, \text{ then} \vdash \Box\psi \to \Box\chi.$$

Logics satisfying **Mon** are called *monotonic* [7]. Neigbourhood frames for monotonic logics satisfy the following weakening of **filter** condition:

supplementation $X, Y \in \tau(x)$ if $X \cap Y \in \tau(x)$.

On the other hand, it is straightforward to observe that $\mathbf{M}_{\mathsf{A}}$ is sound with respect to supplemented frames. Hence, we can conclude that

- the axiomatization consisting of $\mathbf{M}_{\mathsf{A}}$ and those axioms and rules of $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}(\mathsf{E})}$ which do not involve $\Diamond$ gives the minimal monotonic $\mathcal{H}(\mathsf{E})$-logic and that
- every monotonic $\mathcal{H}(\mathsf{E})$-logic axiomatized with a set of pure formulas Γ is complete with respect to the class of supplemented neighborhood frames defined by Γ.

Let us now round off our discussion and ask what is the role of **BG** in neighborhood semantics. After everything we learned about its role in topological spaces, the answer should be easy to guess.

A neighborhood frame is *augmented* if for arbitrary family $\{X_i\}_{i\in I}$ of subsets of I, $\Box_\tau \bigcap_{i\in I} X_i = \bigcap_{i\in I} \Box_\tau X_i$ or, dually, $\Diamond_\tau \bigcup_{i\in I} X_i = \bigcup_{i\in I} \Diamond_\tau X_i$. Just like in case of Alexandroff spaces, it is easily seen to be equivalent to the following formulation:

Fact 6.6 *A neighborhood frame is augmented iff every $\tau(x)$ contains a smallest element.*

The relationship between Kripke frames and augmented neighborhood frames is exactly the same as relationship between quasi-orders and Alexandroff spaces. In other words, there is a category-theoretical equivalence between Kripke frames and augmented neighborhood frames. We refer the reader to [7,8,16] for details.

Theorem 6.7 *A neighborhood frame is augmented iff it admits* **BG***.*

Proof. Exactly the same as the proof of Theorem 3.4 $\Box$

Again, one could look for variations or weakenings of the **BG** rule, analogous to the **BG**$'$ rule discussed in Section 3.

7 Other hybrid languages

At the time of the workshop, it was not immediately clear to us whether our completeness result could be adapted to hybrid languages that lack the global modality, such as $\mathcal{H}(@)$. Recently, an elegant completeness proof for $\mathcal{H}(@)$ was found by Katsuhiko Sano. It is currently being prepared for publication as a separate note [15].

Complete axiomatizations are also obtainable for some *extensions* of $\mathcal{H}(\mathsf{E})$ using the technique of *local definability*. Certain operators not definable in $\mathcal{H}(\mathsf{E})$ are nevertheless "definable at named points". Examples include the derivative operator $\blacklozenge$, which is locally defined by $@_i(\blacklozenge\phi \leftrightarrow \Diamond(\phi \wedge \neg i))$, and the $\downarrow$ binder, which is locally defined by $@_i(\downarrow x.\phi(x) \leftrightarrow \phi(i))$. As explained in [3], whenever an operator is locally definable in a hybrid language such as $\mathcal{H}(\mathsf{E})$, one can add the relevant local definition to the axiomatization of this language, and obtain an axiomatization for the enriched language that is not only complete, but again *complete for arbitrary pure extensions.*

References

[1] Marco Aiello, Johan van Benthem, and Guram Bezhanishvili. Reasoning about space: the modal way. *Journal of Logic and Computation*, 13:889–920, 2003.

[2] Guram Bezhanishvili, Ray Mines, and Patrick J. Morandi. Topo-canonical completions of closure algebras and Heyting algebras. Under submission.

[3] Patrick Blackburn and Balder ten Cate. Pure extensions, proof rules and hybrid axiomatics. *Studia Logica*, 84(2):281–326, 2006. To appear.

[4] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*. Cambridge University Press, Cambridge, UK, 2001.

[5] Balder ten Cate, David Gabelaia, and Dmitry Sustretov. Modal languages for topology: expressivity and definability. Under submission.

[6] Balder ten Cate, Maarten Marx, and Petrucio Viana. Hybrid logics with Sahlqvist axioms. *Logic Journal of the IGPL*, 13(3):293–300, 2005.

[7] Brian F. Chellas. *Modal Logic*. Cambridge University Press, Cambridge Massachusetts, 1980.

[8] Kosta Došen. Duality between modal algebras and neighbourhood frames. *Studia Logica*, 48:219–234, 1989.

[9] George Gargov, Solomon Passy, and Tinko Tinchev. Modal environment for Boolean speculations. In D. Skordev, editor, *Mathematical Logic and its Applications. Proceedings of the Summer School and Conference dedicated to the 80th Anniversary of Kurt Gödel (Druzhba, 1986)*, pages 253–263. Plenum Press, 1987.

[10] Andrey Kudinov. Difference modality in topological spaces. A talk at *Algebraic and Topological Methods in Non-Classical Logics II*, June 2005, Barcelona. Abstract available at `http://atlas-conferences.com/c/a/p/u/76.htm`.

[11] Andrey Kudinov. Topological modal logics with difference modality. In *Advances in Modal Logic*, volume 6, pages 319–332. College Publications, 2006.

[12] Tadeusz Litak. Isomorphism via translation. In *Advances in Modal Logic*, volume 6, pages 333–351. College Publications, 2006.

[13] Janos A. Makowsky and Martin Ziegler. Topological model theory with an interior operator: consistency properties and back-and-forth arguments. *Arch.Math.Logic*, 21(1):37–54, 1981.

[14] J.C.C. McKinsey and Alfred Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[15] Katsuhiko Sano. Topological completeness for hybrid logics without the global modality. In preparation.

[16] Steve K. Thomason. Categories of frames for modal logic. *Journal of Symbolic Logic*, 40:439–442, 1975.

[17] Yde Venema. Derivation rules as anti-axioms in modal logic. *Journal of Symbolic Logic*, 58:1003–1034, 1993.